problem is again of standard least squares form. Flexibility in the application of least squares techniques is therefore substantially enhanced.

RCA Laboratories
Princeton, New Jersey

1. F. B. HILDEBRAND, *Introduction to Numerical Analysis*, McGraw-Hill Book Co., New York, 1956, p. 261–269.
2. F. B. HILDEBRAND, op. cit., p. 287–288.
3. A. S. HOUSEHOLDER, *Principles of Numerical Analysis*, McGraw-Hill Book Co., New York, 1953, p. 221.
4. R. W. HAMMING, *Numerical Methods for Scientists and Engineers*, McGraw-Hill Book Co., New York, 1962, p. 242.
5. NBS APPLIED MATHEMATICS SERIES, No. 9, *Tables of Chebyshev Polynomials $S_n(\alpha)$ and $C_n(\alpha)$*, U. S. Govt. Printing Office, Washington, D. C., 1952; Introduction.

Improved Asymptotic Expansion for the Error Function with Imaginary Argument

By D. van Z. Wadsworth

The well-known asymptotic approximation to the error function can be markedly improved, for the case with imaginary argument, by adding a simple correction term as shown below. The improved analytic approximation was needed in connection with the analysis of spacecraft and ICBM re-entry trajectories.

By definition* the error function with imaginary argument ix where x is real is

$$\text{(1)} \qquad \operatorname{erf}(ix) = i\int_0^x e^{s^2}\,ds = \frac{i}{2}\int_0^{x^2} t^{-1/2}e^t\,dt = \frac{i}{2}\int_L t^{-1/2}e^t\,dt - \frac{\pi^{1/2}}{2}.$$

The branch cut for $t^{-1/2}$ extends along the negative imaginary axis of the t plane and the Riemann sheet is chosen for which $t^{-1/2}$ is positive on the positive real axis. The path of integration L goes from $-\infty$ to x^2 as shown in Figure 1. Repeated partial integration of the infinite integral yields $-i \operatorname{erf}(ix) = E_n(x) + e_n(x)$ where

$$\text{(2)} \qquad E_n(x) = \frac{x^{-1}e^{x^2}}{2}\sum_0^n r_m x^{-2m}$$

is the asymptotic approximation for the interval $(n - \frac{1}{2}) \leqq x^2 < (n + \frac{1}{2})$ and

$$\text{(3)} \qquad e_n(x) = \frac{1}{2}r_{n+1}\int_L t^{-n-3/2}\,e^t\,dt + \frac{i}{2}\pi^{1/2}$$

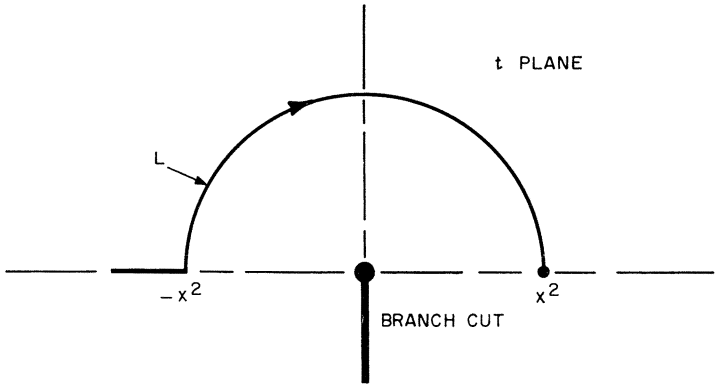
is the error of the asymptotic approximation. The coefficient $r_n = 2^{-2n}(2n)!/n!$.

The integral in equation (3) is equivalent to a line integral on the segment $[-\infty, -x^2]$ and an integral on the semi-circle joining $-x^2$ and x^2. If we let $x^2 \exp(i\pi - i\varphi) = t$ in the latter integral we obtain

$$\text{(4)} \qquad \begin{aligned}\int_L t^{-n-3/2}e^t\,dt = (-)^{n+1}x^{-2n-1}\int_0^\pi \exp\left[-x^2\cos\varphi + ix^2\sin\varphi + i(n+\tfrac{1}{2})\varphi\right]d\varphi \\ + (-)^{n+1}i\int_{x^2}^\infty t^{-n-3/2}e^{-t}\,dt.\end{aligned}$$

Received April 20, 1964.

* This definition differs by a factor of $2\pi^{-1/2}$ from that given by some authors.



PATH OF INTEGRATION

FIGURE 1

If we substitute the right side of equation (4) into equation (3) we obtain

$$(5)\qquad e_n(x) = \frac{r_{n+1}}{2}(-)^{n+1}x^{-2n-1}\int_0^{\pi} e^{-x^2\cos\varphi}\cos\left(x^2\sin\varphi + \left(n+\frac{1}{2}\right)\varphi\right)d\varphi,$$

where we have used the fact that $e_n(x^2)$ is real in order to simplify the right-hand side. (As can be seen from equations (1) and (2), both $-i$ erf (ix) and $E_n(x)$ must be real; consequently $e_n(x)$ must be real.)

To evaluate the integral in equation (5) we expand the cosine factor in a Taylor's series in the independent variable $\sin(\varphi - \pi)$ about the point $\varphi = \pi$:

$$(-)^n\cos\left[x^2\sin\varphi + \left(n+\frac{1}{2}\right)\varphi\right] = \epsilon\sin\varphi + (x^2+\epsilon-\epsilon^3)\frac{\sin^3\varphi}{3!}$$
$$+ ((x^2+\epsilon)(9-10\epsilon^2)+\epsilon^5)\frac{\sin^5\varphi}{5!} + \cdots,$$

where $\epsilon = n + \frac{1}{2} - x^2$.

Next we substitute this expansion into the integrand in equation (5) and employ term-by-term integration, using the fact that the prototype integral with integrand $\exp(-x^2\cos\varphi)\sin^m\varphi$ can be expressed exactly in terms of elementary functions. The final result is

$$(6)\qquad e_n(x) = -\frac{r_{n+1}}{2}e^{x^2}x^{-2n-3}\left[\frac{5}{6} + (n - x^2) + O(x^{-2})\right].$$

If we employ the approximation derived from Stirling's formula, $r_{n+1} \simeq 2^{1/2}(n/e)^n n$ (which is accurate to several per cent even for $n = 1$) we can put equation (6) in the form $e_n(x) = e_n^*(x) + O(x^{-3})$ where

$$e_n^*(x) = -2^{-1/2}x^{-1}(\tfrac{5}{6} + n - x^2).$$

We shall consider $i(E_n(x) + e_n^*(x))$ as our improved asymptotic expansion for the error function with imaginary argument.

In the table, $-i$ erf (ix) as tabulated in Jahnke and Emde's *Tables of Functions* is compared with $E_n(x)$, $E_{n-1}(x)$, and $E_n(x) + e_n^*(x)$. Even at $x^2 = 1$ the improved approximation has only about one per cent error compared to forty per cent for $E_n(x)$.

Accuracy of Asymptotic Approximations

x^2	$-i \operatorname{erf}(x)$	$E_n(x) + e_n^*(x)$	$E_n(x)$	$E_{n-1}(x)$
1.00	1.461	1.449	2.039	1.359
1.25	1.826	1.816	2.185	1.561
1.50	2.250	2.280	3.049	1.830
1.75	2.748	2.750	3.329	2.440
2.00	3.343	3.339	3.755	2.796
2.50	4.935	4.951	5.548	3.865
3.00	7.313	7.310	7.650	6.042
3.50	10.917	10.926	11.430	8.761
4.00	16.450	16.451	16.745	13.419

Bell Telephone Laboratories, Incorporated
Whippany, New Jersey

A One-Step Method for the Numerical Solution of Second Order Linear Ordinary Differential Equations

By J. T. Day

In this paper we shall give a one-step method for the numerical solution of second order linear ordinary differential equations based on Hermitian interpolation and the Lobatto four-point quadrature formula. One-step methods based on quadrature were introduced into the literature by Hammer and Hollingsworth [3]; for subsequent work see Morrison and Stoller [7], and Henrici [5].

Throughout our discussion we shall assume that the functions $N(x), f(x), g(x)$ of the differential equation $y'' = N(x)y' + f(x)y + g(x)$ are sufficiently differentiable to ensure that the derivations we give are valid in any context in which they are used.

In order to simplify somewhat the discussion of the method under consideration we shall first treat the differential equation $y'' = f(x)y + g(x)$, $y(x_0) = y_0$, $y'(x_0) = y_0'$. The necessary modifications for the general second order differential equation $y'' = N(x)y' + f(x)y + g(x)$ will be given later.

After integrating the above differential equation from x_0 to $x_1 = x_0 + h$ $(h > 0)$, we obtain the system of integral equations:

$$(1) \quad y'(x_0 + h) = y'(x_0) + \int_{x_0}^{x_0+h} [f(\tau)y(\tau) + g(\tau)]\, d\tau,$$

Received December 13, 1963.